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On the minimum energy structure of soft, two-dimensional matter in a strong uniform field: 'gravity's rainbow' revisited

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Abstract. Recently, the notion of *conformal crystals* was introduced by Rothen *et al* to describe topologically perfect, two-dimensional (2D) lattices corresponding to locally isotropic deformations of 2D periodic lattices. In the present paper we study the applicability of this idea to classical systems in a uniform field. We prove that within the continuous medium limit *no* conformal mapping of a 2D hexagonal lattice exists which gives any equilibrium structure of an infinitely large 2D system, interacting via a pairwise potential of the form $u(r) = \gamma/r^n$ ($n > 2$), in a uniform gravitational field. The analytical arguments are supported by computer simulations of a system with $n = 3$.

1. Introduction

In many-body systems which are influenced by strong external fields, the density of matter, smoothed over a few intermolecular distances, is position dependent, in general. This is in contrast to bulk structures stable without any external field, like crystals, for which the smoothed density is uniform in space. Thus, it is clear that for systems embedded in external fields, structures of minimum energy cannot be restricted to periodic crystalline lattices. This leads to the question of what are the ordered structures, if any, into which fluids freeze in external fields.

A particular case of this problem was studied experimentally by Pierański [1]. He investigated a 2D system of magnetic spheres placed in a uniform external gravitational field. In the system (consisted of steel balls placed on a slightly tilted plane and interacting by magnetic moments induced by a magnetic field perpendicular to this plane) he discovered a highly ordered structure in which lines connecting subsequent neighbouring particles were not straight, as in the case of periodic crystalline lattices, but formed rainbow-like arcs. The structure observed, and coined by him *gravity's rainbow*, was similar to the structure shown in figure 1. The latter is obtained by mapping sites of a hexagonal lattice contained in a circular stripe by a complex logarithmic function. As is easy to see, this structure exhibits horizontal periodicity.

The characteristic feature of gravity's rainbow is that almost every particle (except those at the surface) has exactly six nearest neighbours, the same as in the hexagonal crystal, which

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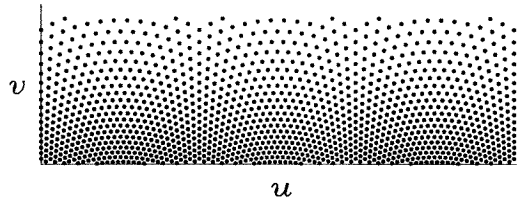

Figure 1. The structure obtained by conformal mapping, $w = (1/\mathrm{i}a)\log az$, of a circular stripe of a hexagonal lattice; $z = x + \mathrm{i}y$, $w = u = \mathrm{i}v$, and (x, y), (u, v) represent the coordinates of the lattice sites of the hexagonal lattice and the conformal lattice, respectively.

is the stable structure of the system without the external gravitational field [1]. In contrast to the zero-gravitational-field crystal, the orientations of the 'atom–atom' bonds in gravity's rainbow depend on the positions of the particles. This allows it to fulfil the condition of non-uniform mass distribution. Visual inspection of the structure recorded by Pierański suggested that the arcs composed of the bonds cross each other at $\pi/3$-angles, the same as in the standard 2D hexagonal lattice. Such an assumption implies that gravity's rainbow can be described by mapping a domain of a hexagonal lattice by a locally angle-preserving transformation. This led to the notion of *conformal crystals* introduced by Rothen *et al* [7] for 2D topologically perfect structures obtained by locally isotropic deformations of (certain subsets, in general, of) 2D periodic lattices. (As is easy to notice, this definition can be generalized to quasicrystalline lattices.) The structure shown in figure 1, as given by an analytic function, is an example of a conformal crystal.

The notion of conformal crystals was expected to be helpful not only in describing freezing of the 2D system of magnetic spheres in the gravitational field [7]. Structures locally similar to gravity's rainbow, although less ordered, were also found in systems of magnetic holes [2] and in crystals grown in special conditions [3]. Ordered arc-like structures also exist in biological systems. The well known example is the problem of phyllotaxis [4], to which an energetic approach has recently been proposed [5, 6].

As follows from the definition, conformal crystals correspond to locally isotropic deformations (i.e. locally pure compression or expansion, without shear) of crystalline lattices. The local stress tensor for such deformations is reduced to pure pressure (see section 4). This feature is characteristic for bulk fluids, and very exceptional for ordered structures when the smoothed density distribution is non-uniform in space. Taking into account the similarity between disordered fluids and the conformal crystals, one might expect that the latter play the role of reference structures for systems freezing in strong fields. Unfortunately, for a given interaction potential of particles forming a system, stable conformal solutions for the structure of the system exist only in particular fields (section 4). Even when these fields do not correspond to situations of practical interest, however, the knowledge of such solutions may be useful *if* modifications (of practical interest) of these fields can lead to minimum-energy structures which correspond to (non-conformal) deformations of the reference conformal structures *without* any change of the topology of the latter. Studies of simple models constitute a way to check if it is so.

Gravity's rainbow is a structure one can start with. Reference [7] may suggest that this structure is an example of a conformal crystal. In the present paper we show that this is not so. Also, our results indicate that, in general, stability of the system of magnetic spheres in the gravitational field requires the presence of defects in the (non-conformally) deformed crystalline lattice.

We studied the problem of gravity's rainbow both numerically, by computer simulations, and analytically, in the limit of a continuous medium. The numerical studies, described in sections 2 and 3, concern finite systems of particles. In section 2 we briefly describe computer simulations performed for systems of some 1500 particles interacting via the 3-inverse-power potential with a hard core. Placing the system in uniform gravitational fields we obtained structures composed of grains with local sixfold symmetry. Although the grains were similar to fragments of gravity's rainbow, the whole structure did not exhibit any horizontal order (periodicity) characteristic of gravity's rainbow. The potential used in the simulations was believed to describe the interparticle interactions in the experimental system of [1]. In section 3 we present a trial to verify this assumption. Our results confirm that the model potential is a reasonable approximation for the interactions of the magnetic spheres. The analytical results (section 4) are obtained for infinite systems, approximated by continuous media. We start with some general considerations concerning conformal crystals and then we illustrate them in the case of a system interacting via the n-inverse-power potential which is embedded in a constant uniform field. We show that, within the continuous medium approximation, no conformal crystal is stable in such a case for any n. We conjecture that no smooth potential can lead to a stable classical conformal crystal in a uniform field. We indicate, however, that the 'logarithmic' conformal crystals can be stable in fields which do not depend on the horizontal coordinate and decay exponentially with the vertical coordinate. The last section contains conclusions.

If not specified explicitly otherwise, all the considerations presented in this paper are restricted to two-dimensional classical systems.

2. Computer simulations

In order to reproduce the experimental results obtained in [1] we studied 2D systems of $N \leqslant 1500$ particles, i.e. close to that investigated experimentally [1]. The advantage of the simulations over the real experiment is that the interaction potential is known exactly and the energy of the system can be directly monitored.

The coordinates of the particles were restricted by

$$0 \leqslant u \leqslant L \qquad 0 \leqslant v$$

where u, v denote the horizontal and vertical coordinates, respectively, and L was used as one of parameters describing the system. The system was embedded in an external gravitational field with the potential:

$$u_{\text{external}} \equiv u_{\text{grav}} = mgv.$$

The particles interacted by a pairwise potential defined as

$$u(r) = \gamma r^{-3} + u_{\text{hard}}(r) \tag{1}$$

where $u_{\text{hard}}(r)$, representing the hard core of the particles of diameter d, is zero for $r \geqslant d$ and infinity elsewhere; for simplicity we took d as unity. Such a choice of $u(r)$ was related to the fact that the above potential was expected to be the simplest model of interactions in the system of magnetic spheres investigated in [1].

As mg only scales the parameter γ in (1) we fixed it to unity, $mg = 1$, and the total energy of the system was given by

$$U_{\text{total}} = \gamma \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} r_{ij}^{-3} + \sum_{i=1}^{N} v_i + U_{\text{hard}}. \tag{2}$$

To search for the minimum of the energy, U_{total}, we used a version of the gradient method in the configurational space. We performed a series of simulations with various γ, L and N, starting from either ordered or disordered structures. We did not succeed in obtaining any structure as ordered as that found by Pierański; one of the typical structures we found is shown in figure 2. As can be seen, this structure consists of some domains of the arc structure with sixfold local symmetry. However, clearly no periodic order in the horizontal direction, present in gravity's rainbow (see figure 2 in [1]), can be detected in figure 2.

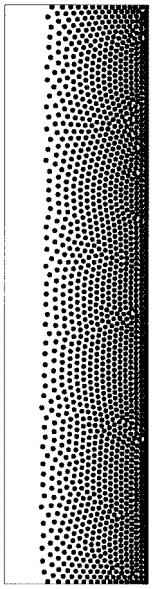

Figure 2. Representative structure for a (locally, at least) stable configuration of the simulated system. It can be seen that domains of the arc structure are separated by defects.

To eliminate the possibility that our lack of success in reproducing gravity's rainbow is an artefact of the numerical procedure used, we also performed some simulations using conformal lattices, similar to that shown in figure 1, as the initial configurations. The structures resulting in the energy minimization process were similar to that shown in figure 2, although they sometimes exhibited slightly better local ordering. We should add here that using a different method of energy minimization, Brownian dynamics with decreasing temperature, one obtains qualitatively the same results also for other systems [8].

Thus, we conclude that the structure, shown in figure 2, can be thought of as representative for a typical minimum energy state of the system studied.

3. Verification of the interaction potential

If one assumes that gravity's rainbow *is not* a universal structure of soft 2D matter in the external uniform field but depends on the details of the interaction potential, one should check how well the 3-inverse-power potential, used in the simulations, approximates the real interactions in [1]. To estimate the 'quality' of the potential applied in the simulations, we used the reasoning described below.

Let us assume that the potential of a single levitating (i.e. distanced from other particles

by more than the core diameter) particle is

$$u_i = \gamma \sum_{j\neq i}^{N} r_{ij}^{-3} + v_i \tag{3}$$

and the total force acting on it is equal to

$$\boldsymbol{F}_i = \left(3\gamma \sum_{j\neq i}^{N}(u_i - u_j)r_{ij}^{-5}, 3\gamma \sum_{j\neq i}^{N}(v_i - v_j)r_{ij}^{-5} - 1\right) \equiv (\gamma f_{u_i}, \gamma f_{v_i} - 1). \tag{4}$$

In equilibrium, the force $\boldsymbol{F}_i$ has to be zero. Obviously, the same must be true for the square of its absolute value, $|\boldsymbol{F}_i|^2$. Taking a group of M levitating particles in equilibrium, we have

$$0 = \|\boldsymbol{F}_M\| \equiv \sum_{i=1}^{M} |\boldsymbol{F}_i|^2 = \sum_{i=1}^{M}[(\gamma f_{u_i})^2 + (\gamma f_{v_i} - 1)^2]. \tag{5}$$

In principle, the above equation could be used to determine γ from the coordinates of the particles. One should take, however, into account that by scanning the positions of the particles some errors are possible, which may cause that, for the scanned coordinates, no solution of (5) exists for γ. Hence, it is better to search for γ which will minimize the norm $\|\boldsymbol{F}_M\|$ in spite of making it zero. (If the components of the particles are known exactly, both ways are equivalent.) From the condition

$$\frac{\partial}{\partial \gamma}\|\boldsymbol{F}_M\| = 0$$

one obtains the required equation for γ

$$\gamma = \frac{\sum_{i=1}^{M} f_{v_i}}{\sum_{i=1}^{M}(f_{u_i}^2 + f_{v_i}^2)}. \tag{6}$$

Using a scanner, we analysed a few photographs of the system of magnetic spheres. The photographs provided us by Piotr Pierański, corresponded to various voltages, U, applied to the coils which were the source of the magnetic field. As the magnetic moment induced in steel balls is proportional to the magnetic field, and hence to the voltage, the parameter γ should be proportional to the square of the voltage. In figure 3 we present the plot of the parameter γ, determined by the above method from the photographs, versus U^2. As can be seen, γ is (roughly) proportional to U^2. Thus, the 3-inverse-power potential can be thought of as a reasonable approximation of the magnetic sphere interactions.

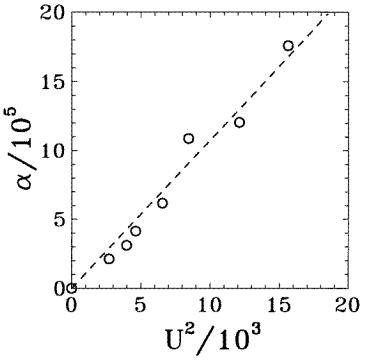

Figure 3. Parameter γ (in arbitrary units), calculated from (6) for a few experimental structures, versus the square of the voltage, U^2. The broken line represents the linear fit to the data; errors are of the order of the size of the circles.

Remark 1. The above method can be easily generalized to potentials composed of many terms with coefficients γ_k ($k = 1, \ldots, m$). In such a case one needs to solve a set of m linear equations with unknown coefficients γ_k. Here we do not present such an analysis of the magnetic sphere interactions because our scanning data were not precise enough to give any conclusive results in this case.

4. Continuous medium approximation

The systems studied above numerically are finite and, hence, subject to boundary conditions which can modify an 'ideal' structure, like gravity's rainbow, one might expect in an infinite system ($L, N \to \infty$, $L/N =$ constant). The latter system is, however, very difficult to study in the discrete case. For this reason we decided to use the continuous medium approximation, for which analytic calculations can be done efficiently. (This approximation is on the same level of idealization as the arguments leading to the concept of the conformal crystals [7].) The calculations within this approximation should help to answer the question if, and when, the stable structure of a system in a uniform external field can correspond to a conformal crystal.

4.1. General considerations

In this subsection we describe some general properties of conformal crystals within the limit of a continuous medium.

Let us denote by

$$w = f(z) \qquad \text{where } z = x + \mathrm{i}y \text{ and } w = u(x, y) + \mathrm{i}v(x, y) \tag{7}$$

an analytic function which transforms points (x_k, y_k) of a certain lattice (in this paper we will assume that this is a hexagonal lattice) on the (x, y)-plane into points (u_k, v_k) of the (u, v)-plane. Further on we will treat the initial (non-deformed) lattice as a continuous elastic medium of given equation of state and elastic properties which are determined by the interactions between the points of the lattice. We will study the properties of the resulting (deformed) continuous medium, expecting that it will reasonably reproduce the properties of the resulting lattice everywhere, except the regions where the density of the system changes rapidly on the length scale of the lattice constant.

The components of the Lagrange strain tensor of a deformation described by the above mapping are [9, 10]:

$$\epsilon_{xx} = [(\partial_x u)^2 + (\partial_x v)^2 - 1]/2 \tag{8}$$

$$\epsilon_{yy} = [(\partial_y u)^2 + (\partial_y v)^2 - 1]/2 \tag{9}$$

$$\epsilon_{xy} = \epsilon_{yx} = (\partial_x u \partial_y u + \partial_x v \partial_y v)/2 \tag{10}$$

where ∂_ζ means differentiation with respect to $\zeta = x, y$.

The Cauchy–Riemann conditions [11] for the analytic function $w = f(z)$:

$$\partial_x u = \partial_y v \tag{11}$$

$$\partial_y u = -\partial_x v \tag{12}$$

imply that the strain tensor is locally isotropic:

$$\epsilon_{xx} = \epsilon_{yy} \tag{13}$$

$$\epsilon_{xy} = \epsilon_{yx} = 0. \tag{14}$$

On the other hand, the Cauchy–Riemann conditions (11), (12) can be obtained from the local isotropy of the strain tensor. Namely, it follows from (14) that

$$\partial_x u \partial_y u = -\partial_x v \partial_y v. \tag{15}$$

Multiplying both sides of (13) by $(\partial_y u)^2$ and using (15) one obtains

$$(\partial_x v)^2(\partial_y v)^2 + (\partial_x v)^2(\partial_y u)^2 = (\partial_y u)^4 + (\partial_y v)^2(\partial_y u)^2$$

what can be rewritten as

$$[(\partial_y u)^2 + (\partial_y v)^2][(\partial_x v)^2 - (\partial_y u)^2] = 0.$$

For any non-isometric transformation this implies either (12) or

$$\partial_x v = \partial_y u. \tag{16}$$

Equation (12) combined with (15) leads to (11). Equations (11), (12) constitute the Cauchy–Riemann conditions for the function $f(z) = u(x, y) + \mathrm{i}v(x, y)$. Equation (16) combined with (15) leads to

$$\partial_x u = -\partial_y v. \tag{17}$$

Equations (16) and (17) constitute the Cauchy–Riemann conditions for the function $\bar{f}(z) = u(x, y) + \mathrm{i}\bar{v}(x, y)$, where $\bar{v}(x, y) \equiv -v(x, y)$.

Thus, the condition that a mapping is conformal is equivalent to the requirement of local isotropy of the strain, i.e. that any infinitesimal volume is uniformly compressed and not sheared [7].

The Jacobian of the transformation $(x, y) \to (u, v)$:

$$j(u, v|x, y) = \partial_x u \partial_y v - \partial_x v \partial_y u = (\partial_x u)^2 + (\partial_y u)^2 = \left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^2$$

gives the ratio of the infinitesimal volumes after and before the transformation. Hence, the relation between the corresponding densities is given by

$$\rho_w = \rho_z \left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^{-2} \tag{18}$$

where $\rho_z =$ constant is the density of the initial (hexagonal) lattice. The assumption that $f(z)$ is an analytic function is then equivalent to the assumption that $\log \rho_w$ is a harmonic function, i.e.

$$\Delta_w \ln \rho_w = 0 \tag{19}$$

where $\Delta_w = (\partial_u)^2 + (\partial_v)^2$. This can be easily shown writing the first derivative of the analytic function:

$$z = f^{-1}(w) \equiv h(w) \tag{20}$$

in the form

$$h'(w) = R(u, v) \exp[\mathrm{i}\phi(u, v)]. \tag{21}$$

The logarithm of $h'(w) \equiv \mathrm{d}z/\mathrm{d}w$ is an analytic function with the real part

$$\log R(u, v) = \tfrac{1}{2} \log (\rho_w/\rho_z) \tag{22}$$

and the imaginary part $\phi(u, v)$. As the real (and the imaginary) part of any analytic function is harmonic [11], one obtains (19). On the other hand, if (19) is fulfilled, then using (22) one can show that $\log R(u, v)$ is a harmonic function which generates the analytic function

$h'(w)$ defined in (21). The latter function is related, via equation (20), to the analytic transformation $w = f(z)$.

Since the infinitesimal volume is uniformly compressed by the conformal deformation, the stress tensor, $\boldsymbol{\sigma}$, at the point (u, v) must be locally isotropic and depends only on the density

$$\boldsymbol{\sigma} = -p[\rho(w)]\boldsymbol{I} \tag{23}$$

where $p(\rho)$ is the pressure corresponding to the density ρ and $\boldsymbol{I}$ is the unit matrix. As the density of the local force field, $\boldsymbol{f} \equiv (f_u, f_v)$, stabilizing the deformation is the minus divergence of the stress [9], $\boldsymbol{f}$ must be equal to the gradient of the pressure:

$$\boldsymbol{f} = -\boldsymbol{\nabla}_w \cdot \boldsymbol{\sigma} = \boldsymbol{\nabla}_w p[\rho(w)] = \frac{\partial p}{\partial \rho_w}\boldsymbol{\nabla}_w \rho_w \tag{24}$$

where $\boldsymbol{\nabla}_w = \hat{\boldsymbol{u}}\partial_u + \hat{\boldsymbol{v}}\partial_v$, and $\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}$ are the unit vectors in the u, v directions, respectively.

Equation (24) gives the density distribution of *any* locally isotropic system whose equation of state is given by $p = p(\rho)$ and which is placed in an external field $\boldsymbol{f}$. (This can be, for example, a fluid!) We should stress that, in general, this solution does not correspond to any conformal mapping of a periodic crystalline lattice. Thus for any conformal mapping of a periodic lattice the density must additionally fulfil equation (19).

4.2. The case of a vertical external field

In this subsection it is shown that all possible conformal crystalline structures in a vertical field correspond to a complex logarithmic mapping [7].

If we assume that the force has only a vertical component, i.e. if we consider a more general case than that of a uniform field, we have

$$0 = f_u = \frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial u}. \tag{25}$$

As the requirement of the stability of the system implies

$$\frac{\partial p}{\partial \rho} > 0. \tag{26}$$

Equation (25) can be replaced by

$$0 = \frac{\partial \rho}{\partial u} = \rho_z \frac{\partial}{\partial u}\left|\frac{\mathrm{d}z}{\mathrm{d}w}\right|^2. \tag{27}$$

Consequently, the absolute value $R \equiv |h'(w)|$ of the analytic function $h'(w) \equiv \mathrm{d}z/\mathrm{d}w$ has to be independent of u, which means that the function can be written as

$$h'(w) = R(v)\exp[\mathrm{i}\phi(u, v)].$$

The real and imaginary parts of this function:

$$\begin{aligned} \xi &= \mathrm{Re}\, h'(w) = R(v)\cos[\phi(u, v)] \\ \eta &= \mathrm{Re}\, h'(w) = R(v)\sin[\phi(u, v)] \end{aligned} \tag{28}$$

have to fulfil the Cauchy–Riemann conditions:

$$\partial_u \xi = \partial_v \eta \tag{29}$$

$$\partial_v \xi = -\partial_u \eta \tag{30}$$

which, in explicit form, can be written as

$$\begin{aligned} &\frac{\partial R}{\partial v}\sin\phi + R\frac{\partial \phi}{\partial u}\sin\phi + R\frac{\partial \phi}{\partial v}\cos\phi = 0 \\ &\frac{\partial R}{\partial v}\cos\phi + R\frac{\partial \phi}{\partial u}\cos\phi - R\frac{\partial \phi}{\partial v}\sin\phi = 0 \end{aligned} \tag{31}$$

and lead to

$$\begin{aligned} &\frac{\mathrm{d}R(v)}{\mathrm{d}v} + R(v)\frac{\partial \phi}{\partial u} = 0 \\ &R(v)\frac{\partial \phi}{\partial v} = 0. \end{aligned} \tag{32}$$

The second equation of (32) implies that ϕ does not depend on v, whereas the first leads to

$$R(v) = R_0 \mathrm{e}^{-v\phi'(u)}.$$

As $R(v)$ does not depend on u one gets

$$\phi(u) = \alpha u + \text{constant}.$$

Thus

$$\frac{\mathrm{d}z}{\mathrm{d}w} \equiv R\mathrm{e}^{\mathrm{i}\phi} = R_0 \mathrm{e}^{\mathrm{i}\times\text{constant}}\mathrm{e}^{-\alpha v + \mathrm{i}\alpha u} = \tilde{R}_0 \mathrm{e}^{\mathrm{i}\alpha w}.$$

Hence

$$z = \frac{\tilde{R}_0}{\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\alpha w}$$

and [7]

$$f(z) \equiv w = \frac{1}{\mathrm{i}\alpha}\log z + \frac{1}{\mathrm{i}\alpha}\log\frac{\mathrm{i}\alpha}{\tilde{R}_0} = \frac{1}{\mathrm{i}\alpha}\log z + f_1 \tag{33}$$

where $f_1 = f(1)$. In the following we will assume that $\alpha > 0$ which corresponds to the field directed opposite to $\hat{\boldsymbol{v}}$.

The above result means that the only conformal mapping fulfilling the requirement that the field is vertical is the logarithmic mapping. For such a mapping one has

$$\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right| = \frac{1}{\alpha|z|} \tag{34}$$

and the density of the conformal crystal is

$$\begin{aligned} \rho_w &= \rho_z|\alpha z|^2 = \rho_z\alpha^2|\exp[\mathrm{i}\alpha(w - f_1)]|^2 \\ &= \rho_z\alpha^2|\exp(-\mathrm{i}\alpha f_1)|^2\exp(-2\alpha v). \end{aligned} \tag{35}$$

It is easy to check that for the mapping given by (33):

$$\frac{\partial \rho_w}{\partial u} = 0 \tag{36}$$

and

$$\frac{\partial \rho_w}{\partial v} = -2\alpha\rho_w. \tag{37}$$

Remark 2. Applying similar considerations to another simple and interesting case, which corresponds to a field of a radial symmetry, one concludes that the class of possible conformal mappings is also very narrow. The only possibilities are [7]

$$w = Cz^{1/\beta} \qquad \text{or} \qquad w = \mathrm{e}^{Cz}$$

where $\beta \neq 0$.

4.3. n-inverse-power potentials

Below we consider a conformal mapping of a system interacting through the n-inverse-power potential. We show that within the continuous medium approach no conformal mapping exists which would transform a perfect hexagonal lattice into a structure stable in the presence of a uniform external field.

Without any external field, the energy per site of an infinite hexagonal lattice (which, for $n > 2$, is believed to correspond to the globally stable structure of the studied 2D system) of particles interacting by the pair-wise potential $u(r) = \gamma r^{-n}$ is equal to

$$U_n(a) = \tfrac{1}{2}\sum_{i\neq 0} u(|\boldsymbol{r}_i|) \tag{38}$$

where $\boldsymbol{r}_i$ denote the positions of the lattice sites of the hexagonal lattice when the centre of the coordinate system is located at $\boldsymbol{r}_0$; a is the lattice constant. Taking into account that

$$u(r) = r^{-n}u(1)$$

one can write

$$U_n(a) = U_n(1)a^{-n} \equiv U_n a^{-n}$$

where U_n is the energy per site of the hexagonal lattice with the lattice constant $a = 1$:

$$\begin{aligned} U_n &\equiv 3\sum_{k=1}^{\infty}\sum_{l=0}^{\infty} u[(k+l/2)^2 + (\sqrt{3}l/2)^2] \\ &= 3\gamma\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}[(k+l/2)^2 + (\sqrt{3}l/2)^2]^{-n/2}. \end{aligned} \tag{39}$$

Differentiation of the energy per site, $U_n(a)$, with respect to the volume per site, $V = \sqrt{3}a^2/2 \equiv \rho^{-1}$, gives the pressure

$$\begin{aligned} p &\equiv -\frac{\partial}{\partial V}U_n(a) = -\frac{1}{\sqrt{3}a}\frac{\partial}{\partial a}(U_n a^{-n}) = 3^{-1/2}nU_n a^{-(n+2)} \\ &= 3^{n/4}2^{-(n+2)/2}nU_n\rho^{1+n/2} \equiv C_n\rho^{1+n/2}. \end{aligned} \tag{40}$$

Substituting equation (40) into (24) and using (36) and (37), one obtains the following result for the components of the gravitational field:

$$g_u = 0 \tag{41}$$

$$\begin{aligned} g_v &= f_v/\rho_w = \frac{\partial p}{\partial \rho}\,\frac{\partial \rho_w}{\partial v}/\rho_w = -(n+2)\alpha C_n\rho_w^{n/2} \\ &= -(n+2)\alpha^{(n+1)}C_n|\exp(-\mathrm{i}n\alpha f_1)|\rho_z^{n/2}\exp(-n\alpha v). \end{aligned} \tag{42}$$

As is easy to see, g_v is an exponentially decreasing function of the height, v, for any positive n. This means that no uniform field can stabilize the perfect conformal crystal with the n-inverse-power potential.

4.4. Elastic properties and stability

In this section we discuss elastic properties and the stability of conformal crystals and illustrate them in the case of the n-inverse-power interactions.

To describe the elastic properties of an isotropic medium two elastic constants are necessary, in general [9, 10]. In the present considerations it will be convenient to use in this role: (i) the bulk modulus, B, measuring the resistance of a material with respect to a

change of its volume (compression) and (ii) the shear modulus, μ, measuring the material resistance with respect to a change of its shape (shear). When the isotropic material is conformally deformed the local deformation corresponds to pure compression and, hence, a single elastic constant (the bulk modulus) is sufficient to describe such a deformation.

If the equation of state of the system is known, the bulk modulus can be easily calculated by differentiating the pressure with respect to the volume:

$$B = -V(\partial p/\partial V) = \rho(\partial p/\partial \rho). \tag{43}$$

Although the calculation of the shear modulus can be non-trivial in general, it is very simple for a 2D static and isotropic crystal, like the hexagonal lattice interacting via the n-inverse power potential. This is so, because in such a case the Cauchy relations hold true [10]. These relations imply that

$$\mu = B/2 - p. \tag{44}$$

Thus, the knowledge of (the pressure and) the bulk modulus is sufficient to describe elastic properties of any 2D static and isotropic crystal and, hence, also the (local) elastic properties of a static conformal crystal.

In the particular case of the static lattice with the central n-inverse-power interactions, the pressure alone is sufficient to determine the elastic constants:

$$B = (n/2 + 1)p \tag{45}$$

$$\mu = (n/4 - 1/2)p. \tag{46}$$

In the general case, however, if the Cauchy relations do not hold (e.g. the interactions are not central, the temperature is positive, etc), one needs *two* independent elastic constants to describe the elastic properties of an isotropic body. The same is true (locally) for conformal crystals if non-conformal deformations are allowed. (Obviously, for conformal deformations a single elastic constant, the bulk modulus, is sufficient.)

Stability of an isotropic system requires that both B and μ are positive. It follows from (46) that for $n \leqslant 2$ both the isotropic as well as any conformal crystal cannot be stable.

4.5. Remark on other potentials

Below we discuss the closure that no smooth potential exists for which a static conformal crystal can be stable in a uniform field.

For the conformal mapping corresponding to a uniform field, it follows from equations (24) and (37) that

$$\rho_w g_v = \frac{\partial p}{\partial \rho}\frac{\partial \rho_w}{\partial v} = \frac{\partial p}{\partial \rho}(-2\alpha)\rho_w. \tag{47}$$

Solving the above equation with $g_v = -g =$ constant, one obtains the following dependence of the pressure on the density:

$$p = \frac{g}{2\alpha}\rho. \tag{48}$$

It is easy to check that the above equation of state is fulfilled for a lattice with a pairwise nearest-neighbour interaction potential which is proportional to the logarithm of the interparticle distance†. However, as was mentioned in the previous section, the system

† Although the interaction range is not well defined for such a potential, the advantage of this choice is that one avoids infinities appearing in the calculations of the energy of the infinite lattice with the potential defined in the standard way: $u_{\log}(r) = c_1 + c_2 \log r$; cf equation (39).

interacting via the logarithmic potential has no stable static configurations in 2D. (This can be easily obtained by explicit calculations, showing that μ is negative, i.e. the stability conditions do not hold. The special form of the logarithmic potential also makes it possible to deduce it from general considerations. The latter is possible because the logarithmic potential is a solution of the Laplace equation in 2D. In such a case the average value of the potential on a circle centred on any point, and not containing a particle, is equal to the value at this point. Thus, no point corresponds to the minimum energy.)

We did not find any potential which led to the equation of state given by (48) on a stable lattice of points interacting through this potential. We expect that such a potential does not exist. Thus, we conjecture that no smooth interparticle pairwise potential exists, for which any conformal crystal is stable in a constant gravitational field.

4.6. Gravity's rainbow

To understand why gravity's rainbow can be observed in the experiment despite the fact that, as follows from the previous sections, it does not correspond to a conformal crystal, it is worthwhile considering a certain class of *non-conformal* deformations of the logarithmic mapping.

Let us consider the following mapping of the (x, y)-plane, represented in the polar coordinates (R, ϕ), into the (u, v)-plane:

$$\begin{aligned} u &= \phi/\alpha \\ v &= V\left(-\frac{1}{\alpha}\log R\right). \end{aligned} \tag{49}$$

As is easy to see, the above mapping corresponds to the (conformal) logarithmic mapping if $V(z) \equiv z$; other choices correspond to non-conformal deformations of the initial lattice.

For the mapping defined by (49), the components of the (local) strain tensor in the (u, v) coordinates are equal to

$$\begin{aligned} \epsilon_{uu} &= [1-(\alpha R)^2]/2 \\ \epsilon_{vv} &= [1-(\alpha R)^2]\Big/2\left[V'\left(-\frac{1}{\alpha}\log R\right)\right]^2 \\ \epsilon_{uv} &= 0. \end{aligned} \tag{50}$$

We should stress here that the local strain tensor is diagonal in the (u, v) coordinates; there is no shear. Moreover, the local strain does not depend on u. Thus, the local stress tensor in the deformed system must also be diagonal and independent of u. In consequence, as follows from (24), the compensating force field† for the deformation defined in (49) must be vertical.

The above remarks indicate that for any external vertical field which does not depend on u, by a proper 'tuning' of the *free* function $V(z)$, one can construct a (non-conformally deformed) structure which will compensate the applied field. In particular, this is possible in the case of a uniform field, and the experimental data of [1] seem to suggest that this is indeed so. We would like to stress, however, that such solutions are possible *only* when the height of the (levitating part of the) system is not large. For a large height, 'tuning' of the locally isotropic structure corresponding to an exponential field must lead to a large

† The explicit form of the compensating force field can be determined for a given $V(z)$ (and vice versa) if the exact dependence of the energy of the periodic crystal on the density and anisotropy is known. For anisotropic structures interacting via the n-inverse-power potentials this dependence can be determined only numerically.

local anisotropy of the strain in the system. This is unfavourable not only because any anisotropy corresponds to an increase of the energy (which, at a given density, is minimal in the isotropic case) but mainly because the underlying hexagonal lattice becomes *unstable* when the anisotropy reaches a certain value. (As one can check, this value is as small as about 10% in the uniformly deformed 3-inverse-power-lattice for certain orientations of the crystalline axes.)

5. Conclusions

Numerical results obtained for a finite system of particles interacting through the 3-inverse-power potential with a hard core indicate that if the system is confined by hard walls and embedded in a uniform field then the gravity's rainbow, i.e. topologically perfect structure, is rather an exception than a rule in the system. Taking into account the analysis performed in section 3, we expect that this conclusion is also valid for the experimental system described in [1], i.e. gravity's rainbow can be observed only in a *small* domain of parameters characterizing the system, and defects are unavoidable if the height of the levitating part of the structure is large enough.

We proved that, within the continuous elasticity limit, no conformal crystal of particles interacting via an n-inverse-power potential can be stable in a uniform gravitational field. This result indicates that even in the domain of parameters for which gravity's rainbow was observed in the experiment, this structure does not correspond to a conformal crystal. In other words there must be some local anisotropy present in the experimentally observed structure. This anisotropy should induce the creation of defects when the sample is large enough. Taking into account that a perfect conformal crystal, which is locally isotropic and, hence, minimizes the local elastic energy, can exist for the n-inverse power potentials only for exponentially decaying fields, one could speculate about the possibility of the existence of structures (in a uniform field) composed of arcs of different lengths. Shorter (in the horizontal direction) arcs might exist near the surface of the system, and longer ones might be observed deeper in the sample.

We conjecture that no smooth interaction pairwise potential can stabilize any static, infinite conformal crystalline structure in a uniform field. The arguments presented in this paper do not exclude, however, the possibility of the existence of some *non-static* conformal structures, i.e. in which the lattice sites correspond only to the probability maxima of finding particles. Such a situation may correspond, for example, to some quantum systems.

The results presented in this work indicate that in the case of vertical external fields the notion of conformal crystals is of rather limited application: for n-inverse-power potentials it can be applied only to exponentially decaying fields which do not play any crucial role in practice. We do not claim, however, that it must be so also for fields of other symmetries. Amongst them, fields of central symmetry are of particular interest. This is related to the fact that various regular patterns of central symmetry exist in biological systems, and the energetic approach to such structures seems to be very promising [5, 6]. Calculations similar to those presented in sections 4.2 and 4.3 show, for example, that conformal crystals can be formed in a field corresponding to a centrifugal force.

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